V. $K^-+p \to \Lambda+\varphi$

The process

$$\begin{array}{l} K^-+p \to \quad \Lambda \quad +\varphi \\ \qquad\qquad\quad \downarrow \qquad\quad \searrow \\ \qquad\qquad \pi^-+p \qquad K+\bar{K} \end{array} \tag{27}$$

is also described by an angular correlation function F defined as in (3). In this case, $\mathbf{k}$ is the momentum of K in the φ rest frame. Because φ is a 1^- particle, the matrix element for $\varphi \to K+\bar{K}$ is also proportional to $\mathbf{k}\cdot\hat{n}$ when the φ spin component along $\hat{n}$ is zero. Therefore, reinterpreting $\mathbf{k}$ accordingly, one may apply all the discussions and results concerning F of the previous sections to (27).

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Proof of a Conjecture of S. Weinberg*

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A proof is given of Weinberg's conjecture[1] that the irreducible N-particle kernel is of Hilbert-Schmidt type if the potentials, which describe the pair interactions, are square integrable.

INTRODUCTION

THIS note merely serves as an appendix to a recent paper by S. Weinberg on multiparticle scattering problems.[1] In the framework of nonrelativistic quantum mechanics, Weinberg has developed an integral equation for the full Green's function of a system of N pairwise-interacting particles which is free of the inadequacies of Lippman-Schwinger type equations. This equation is of the form

$$G(W)=D(W)+I(W)G(W), \tag{1}$$

where W is the complex energy parameter, $D(W)$ the disconnected part of the Green's function $G(W)$, and $I(W)$ an integral operator whose kernel is called the irreducible N-particle kernel. Both $D(W)$ and $I(W)$ are known explicitly in terms of the Green's functions of all subsystems with a smaller number of particles. Weinberg has conjectured that $I(W)$ is of Hilbert-Schmidt type (HS type), if all the pair interactions are described by square-integrable potentials, and he has given a proof for the cases $N=2$, 3. Since his conclusions are based essentially on this conjecture, a proof for arbitrary N is certainly desirable. To furnish such a proof is the only objective of this note.

A NEW REPRESENTATION OF $I(W)$

If not defined otherwise, our terminology is that of Weinberg. We consider an arbitrary decomposition D of the system of particles $(1\cdots N)$ into clusters. The Hamiltonian H_D of this decomposed system is then the full Hamiltonian minus all interactions between different clusters. Weinberg has given an explicit expression for $I(W)$ in terms of all Green's functions $G_D=(W-H_D)^{-1}$. However, in this form $I(W)$ appears as a sum of operators which are not even completely continuous, and the connectedness is not evident. In order to prove that $I(W)$ is of HS type, we have to put it in a different form.

We consider all possible sequences $S=(D_1,D_2,\cdots,D_N)$ of cluster decompositions with the following properties: D_1 is the trivial decomposition into one cluster $(1\cdots N)$, and D_k is obtained from D_{k-1} by splitting one of the clusters of D_{k-1} into two parts. Therefore, each sequence S has N terms and ends with the finest possible decomposition $D_N=(1)(2)\cdots(N)$. By aid of these sequences we can write the connected part $C(W)$ of $G(W)$ in the form

$$C(W)=\sum_{\text{all } S} G_{D_N}V_{D_ND_{N-1}}G_{D_{N-1}}\cdots G_{D_2}V_{D_2D_1}G_{D_1} =I(W)G(W), \tag{2}$$

where $V_{D_kD_{k-1}}$ is the sum of all interactions which are dropped in the transition from $H_{D_{k-1}}$ to H_{D_k}: $V_{D_kD_{k-1}}=H_{D_{k-1}}-H_{D_k}$. Obviously $G_{D_1}=G$, so that one gets from (2) an expression for $I(W)$ simply by omitting the last factors G_{D_1} in the sum. In this form the connectedness of $C(W)$ and $I(W)$ is evident. We do not want to prove here the equivalence of (2) with Weinberg's expression—the reader may easily check it for $N=2$, 3, 4.

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[1] S. Weinberg, Phys. Rev. **133**, B232 (1964).

PROOF THAT $I(W)$ IS OF HS TYPE

We consider a single sequence $S=(D_1\cdots D_N)$ and simplify the notation by writing k instead of a label D_k. The term of $I(W)$ corresponding to S is then

$$G_N V_{N,N-1} G_{N-1}\cdots G_2 V_{2,1}. \tag{3}$$

Note that the indices here denote numbers of clusters—they are not to be confused with Weinbergs labels. G_N, for instance, is the free Green's function in our notation.

Let us consider the product A_k of all factors up to, and including, G_k:

$$A_k = G_N V_{N,N-1} G_{N-1}\cdots G_k \quad (N\geqslant k\geqslant 2). \tag{4}$$

A_k involves only the potentials which are present in H_k. Its kernel in momentum space will therefore contain k δ functions corresponding to the conservation of the momenta of the k clusters. To be more precise and to simplify the notation, let us consider the case $k=3$, $D_k=(1\cdots i)(i+1\cdots j)(j+1\cdots N)$. Then A_k will be of the form

$$(p_{1\ldots N}|A_k|p_1'\ldots_N)=\delta\sum_1^i(p_m-p_m')\delta\sum_{i+1}^j(p_m-p_m')\times\delta\sum_{j+1}^N(p_m-p_m')(p_{1\ldots N}|A_k{}^{(\mathrm{red})}|p_1'\ldots_N). \tag{5}$$

It is natural to consider A_k as an operator on the Hilbert space $\mathfrak{H}_k$ of the decomposed system formed by the k clusters of D_k (without the degrees of freedom corresponding to the motion of the centers of mass of the k clusters). In the previous example $k=3$, for instance, $\mathfrak{H}_k$ would be the space wave functions $\Psi(p_{1\ldots N})$ with the scalar product

$$(\Psi,\phi)=\int dp_{1\ldots N}\delta(\sum_1^i p_m)\delta(\sum_{i+1}^j p_m)\delta(\sum_{j+1}^N p_m)\times\bar{\Psi}(p_{1\ldots N})\phi(p_{1\ldots N}). \tag{6}$$

A_k is an operator of HS type on $\mathfrak{H}_k$, if its HS norm corresponding to the scalar product (6) is finite. In the example $k=3$ this means

$$\|A_k\|^2_{\mathrm{HS}}=\int dp_{1\ldots N}dp_1'\ldots_N\delta(\sum_1^i p_m)\delta(\sum_{i+1}^j p_m)\delta(\sum_{j+1}^N p_m)\times\delta(\sum_1^i p_m')\delta(\sum_{i+1}^j p_m')\delta(\sum_{j+1}^N p_m')\times|(p_{1\ldots N}|A_k{}^{(\mathrm{red})}|p_1'\ldots_N)|^2<\infty. \tag{7}$$

The main theorem, which we shall prove now by induction in k, is the following:

$A_k(W)$ is a HS operator on $\mathfrak{H}_k$ for

$N\geqslant k\geqslant 2$ and for W not on the spectrum of H_k, (8a)

$\|A_k(W)\|_{\mathrm{HS}}\leqslant \mathrm{const}\,|\mathrm{Re}W|^{-1}$ for $\mathrm{Re}W\to-\infty$, (8b)

if (and only if) the potentials are square integrable. Proof: The theorem obviously holds for $k=N$, since $A_N=G_N=$free Green's function and, on $\mathfrak{H}_N$, $\|A_N\|_{\mathrm{HS}}=|W|^{-1}$.

We assume now that the theorem holds for A_k and prove it for A_{k-1}. In order to simplify the notation, we again take $k=3$ and $D_k=(1\cdots i)(i+1\cdots j)(j+1\cdots N)$, $D_{k-1}=(1\cdots j)(j+1\cdots N)$, and we assume that the only interaction between the clusters $(1\cdots i)$, $(i+1\cdots j)$ is due to a force between the particles i, $i+1$—in other words, we only treat a typical term arising from the sum $V_{k,k-1}$. This does not restrict the general validity of the proof, since a finite sum of HS operators is again a HS operator. Thus we obtain:

$$\begin{aligned}(p_{1\ldots N}|A_kV_{k,k-1}|p_1''\ldots_N)&=\int dp_1'\ldots_N\delta[\sum_1^i(p_m-p_m')]\delta[\sum_{i+1}^j(p_m-p_m')]\\&\quad\times\delta[\sum_{j+1}^N(p_m-p_m')](p_{1\ldots N}|A_k{}^{(\mathrm{red})}|p_1'\ldots_N)\\&\quad\times V(q'-q'')\delta(p'-p'')\prod_{m\neq i,i+1}^{1\ldots N}\delta(p_m'-p_m''),\end{aligned}$$

where

$$p'=p_i'+p_{i+1}',\quad q'=\frac{m_{i+1}p_i'-m_ip_{i+1}'}{m_i+m_{i+1}}=\alpha p_i'-\beta p_{i+1}',$$

and similarly for p'', q''. V is the Fourier transform of the potential between the particles i, $i+1$. After integration over the δ functions, one finds a kernel of type (5) with two δ functions left, corresponding to the two clusters $(1\cdots j)(j+1\cdots N)$. The HS norm of this kernel is then given by

$$\begin{aligned}\|A_kV_{k,k-1}\|^2_{\mathrm{HS}}&=\int dp_{1\ldots N}dp_1''\ldots_N\delta(\sum_1^j p_m)\delta(\sum_{j+1}^N p_m)\delta(\sum_1^j p_m'')\\&\quad\times\delta(\sum_{j+1}^N p_m'')|(p_{1\ldots N}|A_k{}^{(\mathrm{red})}|p_1''\ldots_{i-1},\\&\quad p_i+\sum_1^{i-1}(p_m-p_m''),\,p_{i+1}+\sum_{i+2}^j(p_m-p_m''),\,p_i''{}_{+2\ldots N})|^2\\&\quad\times|V[\alpha\sum_1^i(p_m-p_m'')-\beta\sum_{i+1}^j(p_m-p_m'')]|^2.\end{aligned} \tag{9}$$

p_i'' and p_{i+1}'' only occur in the argument of V and in $\delta(\sum_1^j p_m'')$. The integration over these variables is therefore carried out first, yielding

$$\int dp_i''dp_{i+1}''\delta(\sum_1^j p_m'')|V(\cdots)|^2=\int dp_i''|V[a-(\alpha+\beta)p_i'']|^2,$$

a being independent of p_i''. Since $\alpha+\beta=1$, this gives

$$\int dp\,|V(p)|^2=\|V\|^2. \tag{10}$$

The square integrability of V is therefore necessary. The remaining integral in (9) is an integral over $2N-2$ variables only, p_i'' and p_{i+1}'' are missing. But we can write it again as a $2N$-fold integral by introducing new dependent variables which are again denoted by p_i'', p_{i+1}'':

$$\begin{aligned} p_i''&=p_i+\sum_1^{i-1}(p_m-p_m''); \\ p_{i+1}''&=p_{i+1}+\sum_{i+2}^{j}(p_m-p_m''). \end{aligned} \tag{11}$$

In addition, we have to multiply the integrand, by two δ functions enforcing the relations (11). Then we get

$$\begin{aligned} &\|A_kV_{k,k-1}\|^2_{\mathrm{HS}} \\ &=\|V\|^2\int dp_{1\cdots N}dp_1''\cdots_N\delta\Big(\sum_1^j p_m\Big)\delta\Big(\sum_{j+1}^N p_m\Big)\delta\Big(\sum_{j+1}^N p_m''\Big) \\ &\quad\times\delta\Big[\sum_1^i(p_m-p_m'')\Big]\delta\Big[\sum_{i+1}^j(p_m-p_m'')\Big] \\ &\quad\times|(p_{1\cdots N}|A_k{}^{(\mathrm{red})}|p_1''\cdots_N)|^2. \end{aligned} \tag{12}$$

In order to exhibit the connection of the last integral with the HS norm of A_k we introduce the new variables

$$\begin{aligned} u_n&=p_n-\frac{m_n}{M_1}\sum_1^i p_m \quad (n=1\cdots i); \\ u_n&=p_n-\frac{m_n}{M_2}\sum_{i+1}^j p_m \quad (n=i+1\cdots j); \\ u_n&=p_n \quad (n=j+1\cdots N), \end{aligned}$$

where $M_1=m_1+\cdots+m_i$ and $M_2=m_{i+1}+\cdots+m_j$ are the total masses of the clusters $(1\cdots i)$ and $(i+1\cdots j)$. These variables are not independent, since

$$\sum_1^i u_n=\sum_{i+1}^j u_n=0. \tag{13}$$

In addition, we therefore introduce the total momenta of the clusters $(1\cdots i)$ and $(i+1\cdots j)$:

$$P_1=\sum_1^i p_m;\quad P_2=\sum_{i+1}^j p_m,$$

so that

$$dp_{1\cdots N}=\delta\Big(\sum_1^i u_n\Big)\delta\Big(\sum_{i+1}^j u_n\Big)du_{1\cdots N}dP_1dP_2.$$

Defining u_n'', P_1'', P_2'' in the same way, we obtain after integration over $\delta(P_1-P_1'')\delta(P_2-P_2'')$

$$\begin{aligned} \|A_kV_{k,k-1}\|_{\mathrm{HS}}{}^2=\|V\|^2\int dP_1dP_2\delta(P_1+P_2)du_{1\cdots N}du_1''\cdots_N\delta\Big(\sum_1^i u_n\Big)\delta\Big(\sum_{i+1}^j u_n\Big)\delta\Big(\sum_{j+1}^N u_n\Big)\delta\Big(\sum_1^i u_n''\Big)\delta\Big(\sum_{i+1}^j u_n''\Big)\delta\Big(\sum_{j+1}^N u_n''\Big) \\ \times\Big|\Big(u_n+\frac{m_n}{M_1}P_1,\ n=1\cdots i;\ u_n+\frac{m_n}{M_2}P_2,\ n=i+1\cdots j;\ u_{j+1\cdots N}|A_k{}^{(\mathrm{red})}(W) \\ |u_n''+\frac{m_n}{M_1}P_1,\ n=1\cdots i;\ u_n''+\frac{m_n}{M_2}P_2,\ n=i+1\cdots j;\ u_j''{}_{+1\cdots N}\Big)\Big|^2. \end{aligned} \tag{14}$$

For the values of u_n, u_n'' permitted by the δ functions in (14), the last matrix element is equal to

$$(u_{1\cdots N}|A_k{}^{(\mathrm{red})}\Big(W-\frac{P_1{}^2}{2M_1}-\frac{P_2{}^2}{2M_2}\Big)|u_1''\cdots_N). \tag{15}$$

The reason for this is that the potentials involved in A_k are not affected by a common shift in the velocities of all particles belonging to the same cluster—the shift in the total energy is then due entirely to the change in kinetic energy. The δ functions in (14) imply that the momenta of the three clusters $(1\cdots i)$, $(i+1\cdots j)$, $(j+1\cdots N)$ are zero in the states $|u_{1\cdots N})$ and $|u_{1\cdots N}'')$. Therefore, if all the velocities in these clusters are shifted by P_1/M_1, P_2/M_2, 0, respectively, the total energy changes by an amount $P_1{}^2/2M_1+P_2{}^2/2M_2$. Insertion of (15) in (14), integration over $\delta(P_1+P_2)$ and comparison with (7) yields the desired relation

$$\begin{aligned} \|A_k(W)V_{k,k-1}\|^2_{\mathrm{HS}}&=\|V\|^2 4\pi\int_0^\infty dpp^2 \\ &\quad\times\Big|\Big|A_k\Big(W-\frac{M_1+M_2}{2M_1M_2}p^2\Big)\Big|\Big|^2_{\mathrm{HS}}. \end{aligned} \tag{16}$$

According to our induction assumption the integrand is bounded by const$\times p^{-2}$ for $p\to\infty$, so that the integral converges for $\arg(W-W_0)\neq 0$, W_0 being the lower end of the spectrum of H_k:

$$\|A_k(W)V_{k,k-1}\|^2_{\mathrm{HS}}<\infty \quad \text{for} \quad \arg(W-W_0)\neq 0. \tag{17}$$

In addition we conclude from (16) that

$$\|A_k(W)V_{k,k-1}\|^2_{\mathrm{HS}} \to 0 \quad \text{for} \quad \mathrm{Re}W \to -\infty . \quad (18)$$

The product AB of a HS operator A with a bounded operator B is again a HS operator whose HS norm satisfies $\|AB\|_{\mathrm{HS}} \leqslant \|A\|_{\mathrm{HS}}\|B\|$, $\|B\|$ being the operator norm of B. Therefore

$$\|A_{k-1}(W)\|_{\mathrm{HS}} \leqslant \|A_k(W)V_{k,k-1}\|_{\mathrm{HS}}\|G_{k-1}(W)\| . \quad (19)$$

But $G_{k-1}(W)$ is the resolvent of the self-adjoint[2] operator H_{k-1} on $\mathfrak{H}_{k-1}$ which is bounded from below.[1] Therefore $G_{k-1}(W)$ is bounded whenever W is not on the spectrum of H_{k-1}, and $\|G_{k-1}(W)\| \leqslant \mathrm{const}\,|\mathrm{Re}W|^{-1}$ for $\mathrm{Re}W \to -\infty$. Since the spectrum of H_{k-1} contains the set $\arg(W-W_0)=0$ (in its continuous part), we conclude from (17), (18), and (19) that $A_{k-1}(W)$ has indeed the properties (8a), (8b), so that our induction proof is completed.

For $k=2$, theorem (8) and (17) yield the final result: $I(W)$ is a Hilbert-Schmidt operator on the space $\mathfrak{H}_1$ for all W not on the continuous spectrum of the full Hamiltonian, if (and only if) the potentials, which describe the pair interactions, are square integrable.

This is identical with Weinberg's conjecture, since the HS norm for $\mathfrak{H}_1$ is the same as Weinberg's "center-of-mass HS norm."

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One-Nucleon Exchange in Pion-Nucleon Scattering*

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We have studied the partial-wave amplitudes for pion-nucleon scattering in the approximation where the "driving force" comes from single-nucleon exchange. We start from the assumption that the amplitudes satisfy dispersion relations, but find that this is not enough to define a unique problem. Some restrictions upon the choice of amplitude to be inserted in the dispersion relations are found, but some ambiguities remain. Settling these in a "reasonable" way leads to the conclusion that the one-nucleon-exchange force is "too strong."

I. INTRODUCTION

IT has long been part of the folklore of physics that the resonance observed in the $T=\frac{3}{2}$, $J=\frac{3}{2}^+$ scattering state of the pion-nucleon system is induced by the exchange of a single nucleon in the "crossed" pion-nucleon scattering channel. This belief very likely originated in the static-model calculations of Chew,[1] where it was found that a two-parameter (coupling constant and high-energy cutoff) fit could reproduce the position and width of the observed resonance. The belief was strengthened by the fact that the coupling constant determined by the two-parameter fit agreed with the one obtained subsequently from the forward-scattering dispersion relations.[2] In this way the single-nucleon-exchange diagram was understood to provide the primary driving force in the 3, 3 scattering amplitude.

One would like to adopt the viewpoint that the static model is an approximation to a more complete, relativistic theory which would permit calculations free of arbitrarily imposed cutoff parameters. The discovery of the Mandelstam representation,[3] from which partial-wave dispersion relations were deduced, appeared to provide a means of doing this. Frautschi and Walecka[4] investigated this approach and were able to show that the single-nucleon-exchange force is more than sufficiently attractive to account for the 3, 3 resonance.

Following the qualitative success of the Frautschi-Walecka calculation, it was possible to feel that one had a reasonable understanding of pion-nucleon scattering over a substantial range of energies. For example, if the expression for the single-nucleon-exchange force is reduced to nonrelativistic form, a Yukawa potential is obtained with the sign $(-1)^{I+J+T}$ (a positive sign corresponding to an attractive interaction). This is roughly in agreement with what one is led to deduce from the observed scattering.

The next step was to seek a sharper understanding by carrying through a more precise version of the

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